Governing Equations of Flow and Sediment Transport

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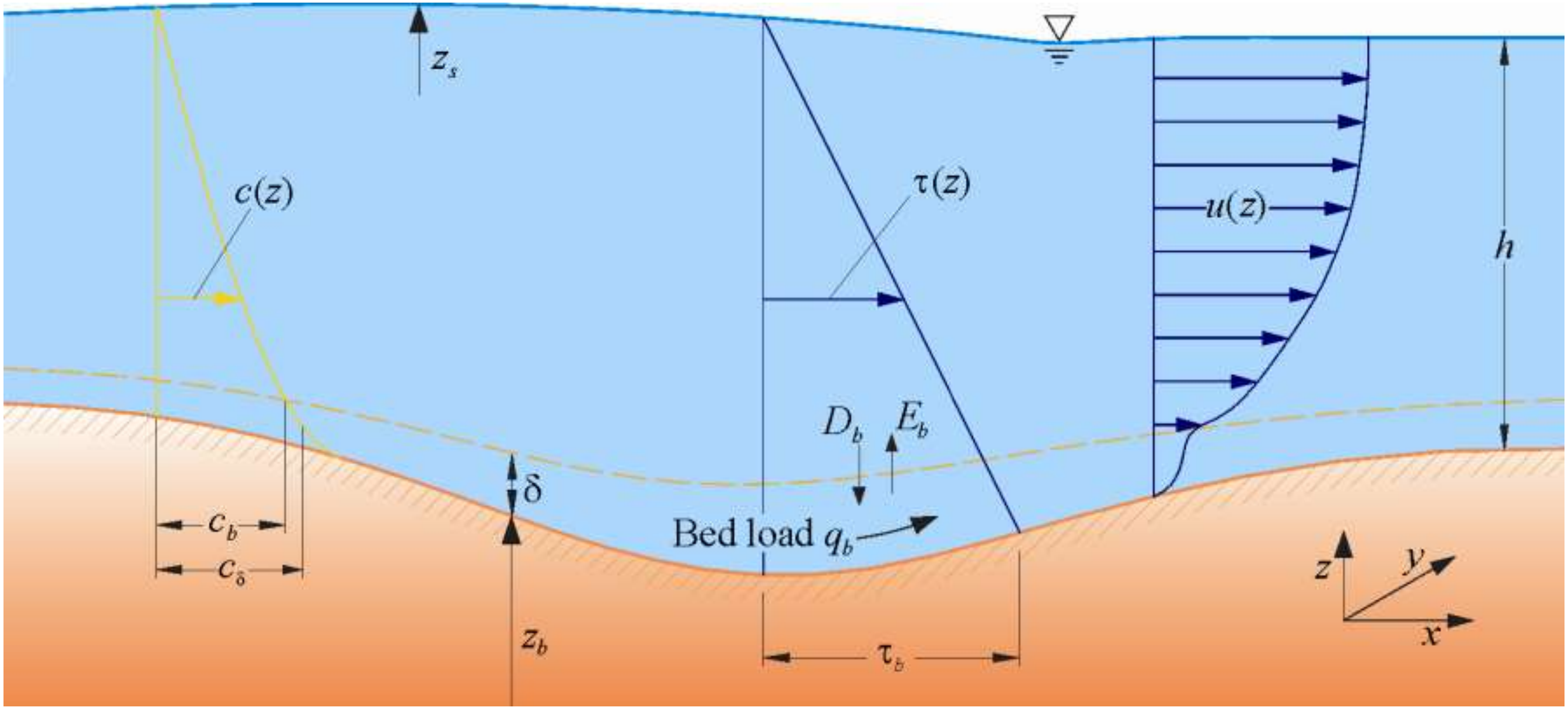
Sketch of Flow and Sediment Transport



3-D Hydrodynamic Equations

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0$$

$$\frac{\partial u_x}{\partial t} + \frac{\partial (u_x^2)}{\partial x} + \frac{\partial (u_y u_x)}{\partial y} + \frac{\partial (u_z u_x)}{\partial z} = \frac{1}{\rho} F_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{xx}}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{xy}}{\partial y} + \frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial z}$$

$$\frac{\partial u_y}{\partial t} + \frac{\partial (u_x u_y)}{\partial x} + \frac{\partial (u_y^2)}{\partial y} + \frac{\partial (u_z u_y)}{\partial z} = \frac{1}{\rho} F_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{1}{\rho}\frac{\partial \tau_{yx}}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{yy}}{\partial y} + \frac{1}{\rho}\frac{\partial \tau_{yz}}{\partial z}$$

$$\frac{\partial u_z}{\partial t} + \frac{\partial (u_x u_z)}{\partial x} + \frac{\partial (u_y u_z)}{\partial y} + \frac{\partial (u_z^2)}{\partial z} = \frac{1}{\rho} F_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{zy}}{\partial y} + \frac{1}{\rho}\frac{\partial \tau_{zz}}{\partial z}$$

Hydrostatic Pressure Assumption

For gradually varied open-channel flows, ignoring the inertia and diffusion effects in the vertical momentum equation yields

$$\frac{\partial p}{\partial z} = -\rho g$$

In cases of a constant density of water:

$$p = p_a + \rho g (z_s - z)$$

Thus, the horizontal momentum equations become:

$$\frac{\partial u_x}{\partial t} + \frac{\partial (u_x^2)}{\partial x} + \frac{\partial (u_y u_x)}{\partial y} + \frac{\partial (u_z u_x)}{\partial z} = -g \frac{\partial z_s}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{xx}}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{xy}}{\partial y} + \frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial z}$$

$$\frac{\partial u_y}{\partial t} + \frac{\partial (u_x u_y)}{\partial x} + \frac{\partial (u_y^2)}{\partial y} + \frac{\partial (u_z u_y)}{\partial z} = -g \frac{\partial z_s}{\partial y} + \frac{1}{\rho}\frac{\partial \tau_{yx}}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{yy}}{\partial y} + \frac{1}{\rho}\frac{\partial \tau_{yz}}{\partial z}$$

Boundary Conditions of Flow

The channel bed and banks generally vary in much slower speed than the flow, so the non-slip condition is applied there:

$$u_{bx} = 0, \quad u_{by} = 0, \quad u_{bz} = 0$$

The water surface is described by

$$z = z_s(x, y, t)$$

A particle at (x, y, z) on the water surface has

$$\frac{dx}{dt} = u_{hx} \qquad \frac{dy}{dt} = u_{hy} \qquad \frac{dz}{dt} = u_{hz}$$

Derivation of the function $z=z_s(x, y, t)$ with respect to t yields the free-surface kinematic condition:

$$\frac{\partial z_s}{\partial t} + u_{hx}\frac{\partial z_s}{\partial x} + u_{hy}\frac{\partial z_s}{\partial y} = u_{hz}$$

Turbulence Closure

The often used turbulence closure models are based on Boussinesq's eddy viscosity concept:

$$\tau_{ij} = \rho \nu_t \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} k \delta_{ij}$$

- **Zero-equation turbulence models**
  - Mixing length model
  - Subgrid model

- **Two-equation turbulence models**
  - Standard k-ε turbulence model
  - RNG k-ε turbulence model
  - Nonequilibrium k-ε turbulence model
  - k-ω turbulence model

- **Other advanced models**: Non-linear k-ε turbulence model, Reynolds stress/flux model, algebraic Reynolds stress/flux model, LES, DNS, etc.

▪Two-Equation Turbulence Models

In the k-ε turbulence model, the eddy viscosity is determined:

$$\nu_t = c_\mu \frac{k^2}{\varepsilon}$$

The modeled *k*-equation:

$$\frac{\partial k}{\partial t} + \frac{\partial}{\partial x_i}(\bar{u}_i k) = \frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_j}\right) + P_k - \varepsilon$$

and the modeled *ε*-equation:

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial}{\partial x_i}(\bar{u}_i \varepsilon) = \frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_j}\right) + \frac{\varepsilon}{k}(c_{\varepsilon 1} P_k - c_{\varepsilon 2}\varepsilon)$$

The coefficients of the standard, non-equilibrium, and RNG k-ε turbulence models are listed in Table 2.3.

k-ε Model	c_μ	$c_{\varepsilon 1}$	$c_{\varepsilon 2}$	σ_k	σ_ε
Standard	0.09	1.44	1.92	1.0	1.3
Non-equilibrium	0.09	$1.15+0.25P_k/\varepsilon$	1.90	0.8927	1.15
RNG	0.085	$1.42-\eta(1-\eta/\eta_0)/(1+\beta\eta^3)$	1.68	0.7179	0.7179

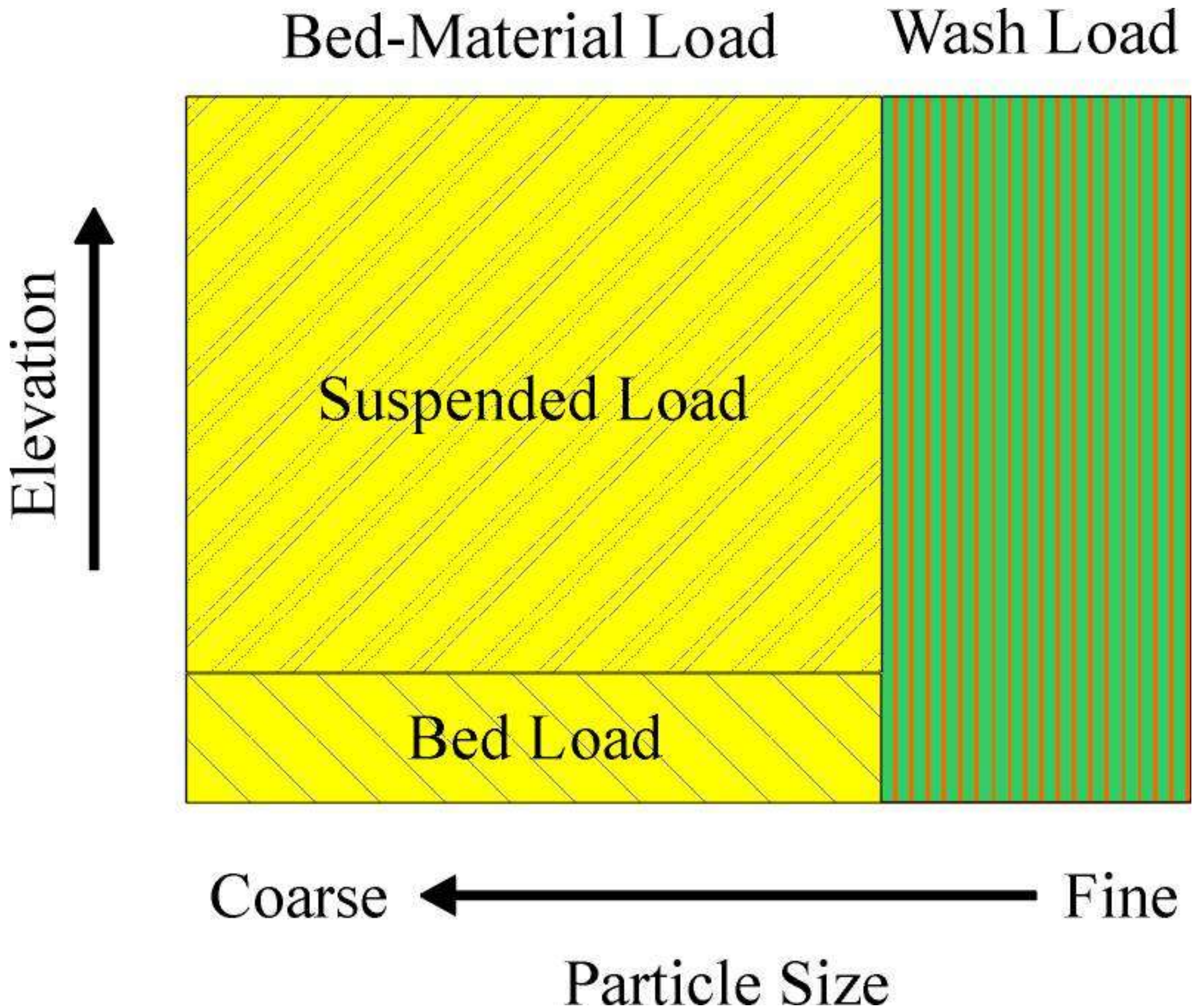
The k-ω turbulence model is also often used, which defines

$$\omega = \varepsilon/(\beta^* k) \qquad \text{with} \qquad \beta^* = 0.09$$

▪Other Turbulence Models and Simulations

- Nonlinear k-ε Turbulence Models
- Algebraic Reynolds Stress/Flux Model
- Reynolds Stress/Flux Models
- Large Eddy Simulation (LES)
- Direct Numerical Simulation (DNS)

Sediment Transport Modes



3-D Sediment Transport Equation

Sediment transport equation

$$\frac{\partial c}{\partial t}+\frac{\partial(u_x c)}{\partial x}+\frac{\partial(u_y c)}{\partial y}+\frac{\partial(u_z c)}{\partial z}-\frac{\partial(\omega_s c)}{\partial z}=\frac{\partial}{\partial x}\left(\varepsilon_s\frac{\partial c}{\partial x}\right)+\frac{\partial}{\partial y}\left(\varepsilon_s\frac{\partial c}{\partial y}\right)+\frac{\partial}{\partial z}\left(\varepsilon_s\frac{\partial c}{\partial z}\right)$$

At water surface,

$$\left(\varepsilon_s\frac{\partial c}{\partial z}+\omega_s c\right)_{z=z_s}=0$$

At the interface between bed load and suspended load, a concentration (at equilibrium condition) may be specified:

$$c\big|_{z=z_b+\delta}=c_{b*}$$

but more generally, the entrainment and deposition fluxes at the interface are specified as

$$E_b=-\varepsilon_s\frac{\partial c}{\partial z}\bigg|_{z=z_b+\delta}=\omega_s c_{b*}\qquad D_b=\omega_s c_b$$

Depth-averaged 2-D Flow Equations

Define the depth-averaged quantity:

$$\Phi = \frac{1}{h}\int_{z_b}^{z_s} \phi \, dz$$

Depth-averaging the continuity equation

$$\int_{z_b}^{z_s} \frac{\partial u_x}{\partial x} dz + \int_{z_b}^{z_s} \frac{\partial u_y}{\partial y} dz + \int_{z_b}^{z_s} \frac{\partial u_z}{\partial z} dz = 0$$

The Leibniz integral rule

$$\frac{\partial}{\partial x}\int_{z_b}^{z_s} u_x dz - u_{hx}\frac{\partial z_s}{\partial x} + u_{bx}\frac{\partial z_b}{\partial x} + \frac{\partial}{\partial y}\int_{z_b}^{z_s} u_y dz - u_{hy}\frac{\partial z_s}{\partial y} + u_{by}\frac{\partial z_b}{\partial y} + u_{hz} - u_{bz} = 0$$

Application of boundary conditions yields

$$\frac{\partial h}{\partial t} + \frac{\partial (hU_x)}{\partial x} + \frac{\partial (hU_y)}{\partial y} = 0$$

Depth-averaging the *x*-momentum equation:

$$\int_{z_b}^{z_s}\frac{\partial u_x}{\partial t}dz+\int_{z_b}^{z_s}\frac{\partial (u_x^2)}{\partial x}dz+\int_{z_b}^{z_s}\frac{\partial (u_y u_x)}{\partial y}dz+\int_{z_b}^{z_s}\frac{\partial (u_z u_x)}{\partial z}dz$$

$$=-g\int_{z_b}^{z_s}\frac{\partial z_s}{\partial x}dz+\frac{1}{\rho}\int_{z_b}^{z_s}\frac{\partial \tau_{xx}}{\partial x}dz+\frac{1}{\rho}\int_{z_b}^{z_s}\frac{\partial \tau_{xy}}{\partial y}dz+\frac{1}{\rho}\int_{z_b}^{z_s}\frac{\partial \tau_{xz}}{\partial z}dz$$

The Leibniz integral rule

$$\frac{\partial}{\partial t}\left(\int_{z_b}^{z_s}u_x dz\right)-u_{hx}\frac{\partial z_s}{\partial t}+u_{bx}\frac{\partial z_b}{\partial t}+\frac{\partial}{\partial x}\left(\int_{z_b}^{z_s}u_x^2 dz\right)-u_{hx}^2\frac{\partial z_s}{\partial x}+u_{bx}^2\frac{\partial z_b}{\partial x}$$

$$+\frac{\partial}{\partial y}\left(\int_{z_b}^{z_s}u_y u_x dz\right)-u_{hy}u_{hx}\frac{\partial z_s}{\partial y}+u_{by}u_{bx}\frac{\partial z_b}{\partial y}+u_{hz}u_{hx}-u_{bz}u_{bx}$$

$$=-gh\frac{\partial z_s}{\partial x}+\frac{1}{\rho}\frac{\partial}{\partial x}\left(\int_{z_b}^{z_s}\tau_{xx}dz\right)-\frac{1}{\rho}\tau_{xx,s}\frac{\partial z_s}{\partial x}+\frac{1}{\rho}\tau_{xx,b}\frac{\partial z_b}{\partial x}$$

$$+\frac{1}{\rho}\frac{\partial}{\partial y}\left(\int_{z_b}^{z_s}\tau_{xy}dz\right)-\frac{1}{\rho}\tau_{xy,s}\frac{\partial z_s}{\partial y}+\frac{1}{\rho}\tau_{xy,b}\frac{\partial z_b}{\partial y}+\frac{1}{\rho}(\tau_{xz,s}-\tau_{xz,b})$$

Application of boundary conditions yields

$$\frac{\partial(hU_x)}{\partial t}+\frac{\partial(hU_x^2)}{\partial x}+\frac{\partial(hU_yU_x)}{\partial y}=-gh\frac{\partial z_s}{\partial x}+\frac{1}{\rho}\frac{\partial[h(T_{xx}+D_{xx})]}{\partial x}$$
$$+\frac{1}{\rho}\frac{\partial[h(T_{xy}+D_{xy})]}{\partial y}+\frac{1}{\rho}(\tau_{sx}-\tau_{bx})$$

Similarly, the depth-averaged *y*-momentum equation is derived as

$$\frac{\partial(hU_y)}{\partial t}+\frac{\partial(hU_xU_y)}{\partial x}+\frac{\partial(hU_y^2)}{\partial y}=-gh\frac{\partial z_s}{\partial y}+\frac{1}{\rho}\frac{\partial[h(T_{yx}+D_{yx})]}{\partial x}$$
$$+\frac{1}{\rho}\frac{\partial[h(T_{yy}+D_{yy})]}{\partial y}+\frac{1}{\rho}(\tau_{sy}-\tau_{by})$$

Clarkson UNIVERSITY defy convention

Dispersion momentum transports

$$D_{xx} = -\frac{\rho}{h}\int_{z_b}^{z_s} (u_x - U_x)^2 dz$$

$$D_{xy} = D_{yx} = -\frac{\rho}{h}\int_{z_b}^{z_s} (u_x - U_x)(u_y - U_y) dz$$

$$D_{yy} = -\frac{\rho}{h}\int_{z_b}^{z_s} (u_y - U_y)^2 dz$$

which are usually combined with the turbulent stresses T_{xx}, T_{xy}, T_{yx}, and T_{yy} in nearly straight channels. In curved channels, these dispersion transports need to be modeled additionally.

Depth-averaged 2-D Sediment Equations

Define the depth-averaged 2-D suspended-load concentration

$$C = \frac{1}{(h-\delta)U_s}\int_{z_b+\delta}^{z_s} u_s c\, dz \qquad \text{I}$$

Depth-averaging the sediment transport equation:

$$\int_{z_b+\delta}^{z_s}\frac{\partial c}{\partial t}dz + \int_{z_b+\delta}^{z_s}\frac{\partial (u_x c)}{\partial x}dz + \int_{z_b+\delta}^{z_s}\frac{\partial (u_y c)}{\partial y}dz + \int_{z_b+\delta}^{z_s}\frac{\partial (u_z c)}{\partial z}dz - \int_{z_b+\delta}^{z_s}\frac{\partial (\omega_s c)}{\partial z}dz$$

$$= \int_{z_b+\delta}^{z_s}\left[\frac{\partial}{\partial x}\left(\varepsilon_s\frac{\partial c}{\partial x}\right)\right]dz + \int_{z_b+\delta}^{z_s}\left[\frac{\partial}{\partial y}\left(\varepsilon_s\frac{\partial c}{\partial y}\right)\right]dz + \int_{z_b+\delta}^{z_s}\left[\frac{\partial}{\partial z}\left(\varepsilon_s\frac{\partial c}{\partial z}\right)\right]dz$$

Use of the Leibniz integral rule and boundary conditions yields

$$\frac{\partial}{\partial t}\left(\frac{hC}{\beta_s}\right) + \frac{\partial (hU_x C)}{\partial x} + \frac{\partial (hU_y C)}{\partial y} = \frac{\partial}{\partial x}\left[h\left(\varepsilon_s\frac{\partial C}{\partial x} + D_{sx}\right)\right] + \frac{\partial}{\partial y}\left[h\left(\varepsilon_s\frac{\partial C}{\partial y} + D_{sy}\right)\right] + E_b - D_b$$

The correction factor β_s is

$$\beta_s = \int_{z_b+\delta}^{z_s} u_s c\, dz \Big/ \left(U_s \int_{z_b+\delta}^{z_s} c\, dz \right)$$

Because $\beta_s < 1$, it accounts for the temporal lag between flow and suspended-load transport.

Dispersion fluxes of suspended load:

$$D_{sx} = -\frac{1}{h} \int_{z_b}^{z_s} (u_x - U_x)(c - C) dz$$

$$D_{sy} = -\frac{1}{h} \int_{z_b}^{z_s} (u_y - U_y)(c - C) dz$$

If the depth-averaged 2-D suspended-load concentration is defined as

$$C = \frac{1}{(h-\delta)} \int_{z_b+\delta}^{z_s} c\, dz \qquad \text{II}$$

The following depth-averaged 2-D equation is obtained:

$$\frac{\partial (hC)}{\partial t} + \frac{\partial (\beta_s h U_x C)}{\partial x} + \frac{\partial (\beta_s h U_y C)}{\partial y} = \frac{\partial}{\partial x}\left[h\left(\varepsilon_s \frac{\partial C}{\partial x} + D'_{sx} \right)\right] + \frac{\partial}{\partial y}\left[h\left(\varepsilon_s \frac{\partial C}{\partial y} + D'_{sy} \right)\right] + E_b - D_b$$

However, corresponding to definition II, the suspended-load discharge should be defined as $q_s = \beta_s UhC$, rather than $q_s = UhC$.

Note that definition I is used here.

The bed-load mass balance equation:

$$(1-p'_m)\frac{\partial z_b}{\partial t}+\frac{\partial(\delta c_\delta)}{\partial t}+\frac{\partial(\alpha_{bx}q_b)}{\partial x}+\frac{\partial(\alpha_{by}q_b)}{\partial y}=D_b-E_b$$

Using relation $c_\delta = q_b/(\delta u_b)$ yields

$$(1-p'_m)\frac{\partial z_b}{\partial t}+\frac{\partial}{\partial t}\left(\frac{q_b}{u_b}\right)+\frac{\partial(\alpha_{bx}q_b)}{\partial x}+\frac{\partial(\alpha_{by}q_b)}{\partial y}=D_b-E_b$$

where u_b is the bed-load velocity. Because u_b is slower than the flow velocity, the above equation accounts for the temporal lag between flow and bed-load transport.

Summation of bed-load and suspended-load transport equations yields the total-load transport equation:

$$(1-p'_m)\frac{\partial z_b}{\partial t}+\frac{\partial}{\partial t}\left(\frac{hC_t}{\beta_t}\right)+\frac{\partial q_{tx}}{\partial x}+\frac{\partial q_{ty}}{\partial y}=0$$

with

$$q_{tx}=\alpha_{bx}q_b+hU_xC-\varepsilon_s h\partial C/\partial x-hD_{sx}$$

$$q_{ty}=\alpha_{by}q_b+hU_yC-\varepsilon_s h\partial C/\partial y-hD_{sy}$$

$$\beta_t=\frac{hC_t}{hC/\beta_s+q_b/u_b}=\frac{1}{r_s/\beta_s+(1-r_s)U/u_b}$$

Exchange Flux of Suspended Load Near Bed

Depth-averaged 2-D suspended-load transport equation

$$\frac{\partial}{\partial t}\left(\frac{hC}{\beta_s}\right)+\frac{\partial(hU_xC)}{\partial x}+\frac{\partial(hU_yC)}{\partial y}=\frac{\partial}{\partial x}\left[h\left(\varepsilon_s\frac{\partial C}{\partial x}+D_{sx}\right)\right]+\frac{\partial}{\partial y}\left[h\left(\varepsilon_s\frac{\partial C}{\partial y}+D_{sy}\right)\right]+E_b-D_b$$

The near-bed deposition flux $D_b=\omega_s c_b$ needs to be modeled, because c_b is not solved in a depth-averaged 2-D (1-D) model.

The near-bed concentration c_b is often related to the depth-averaged concentration C by $c_b = \alpha_c C$, so

$$D_b-E_b=\alpha_c\omega_s C-\omega_s c_{b*}$$

where α_c is the adaptation (recovery) coefficient, and c_{b*} is determined by an empirical formula.

Use of the Rouse distribution yields (Minh Duc, 1998)

$$\alpha_c = (h-\delta) \Big/ \int_\delta^h \left(\frac{h-z}{z} \frac{\delta}{h-\delta} \right)^{\omega_s/\kappa U_*} dz$$

Lin (1984) proposed

$$\alpha_c = 3.25 + 0.55 \ln\left(\frac{\omega_s}{\kappa U_*} \right)$$

which is used by Spasojevic and Holly (1990).

If the near-bed equilibrium concentration c_{b*} is related to the depth-averaged equilibrium concentration C_* by $c_{b*} = \alpha_{c*} C_*$, then

$$D_b - E_b = \alpha_c \omega_s C - \alpha_{c*} \omega_s C_*$$

where α_{c*} is the adaptation (recovery) coefficient under the equilibrium condition, and C_* is determined by an empirical formula.

In equilibrium, $\alpha_{c*} = \alpha_c$; in non-equilibrium, $\alpha_{c*} \neq \alpha_c$.

Because equilibrium is acquired through exchange between bed material and moving sediment near the bed, usually for erosion $C/c_b \leq C_*/c_{b*}$ and $\alpha_c \geq \alpha_{c*}$; for deposition $C/c_b \geq C_*/c_{b*}$ and $\alpha_c \leq \alpha_{c*}$.

The difference between α_c and α_{c*} is often assumed to be negligible, for simplicity. Thus, the net exchange flux can be determined by (Han, 1980; Wu, 1991)

$$D_b - E_b = \alpha\omega_s(C - C_*)$$

where α is a new adaptation coefficient.

From $\alpha\omega_s(C - C_*) = \alpha_c\omega_s C - \alpha_{c*}\omega_s C_*$, one can derive

$$\alpha = \alpha_c + (\alpha_c - \alpha_{c*})\frac{C_*}{C - C_*} \qquad\qquad \alpha = \alpha_{c*} + (\alpha_c - \alpha_{c*})\frac{C}{C - C_*}$$

For erosion, usually $C/c_b \le C_*/c_{b*}$ and $\alpha_c \ge \alpha_{c*}$, thus $\alpha \le \alpha_c$ and $\alpha \le \alpha_{c*}$. For deposition, usually $C/c_b \ge C_*/c_{b*}$ and $\alpha_c \le \alpha_{c*}$, thus $\alpha \le \alpha_c$ and $\alpha \le \alpha_{c*}$.

Armanini and de Silvio's function:

$$\frac{1}{\alpha} = \frac{a}{h} + \left(1 - \frac{a}{h}\right)\exp\left[-1.5\left(\frac{a}{h}\right)^{-1/6} \frac{\omega_s}{U_*}\right]$$

where a is thickness of the bottom layer.

Zhou and Lin (1998) proposed

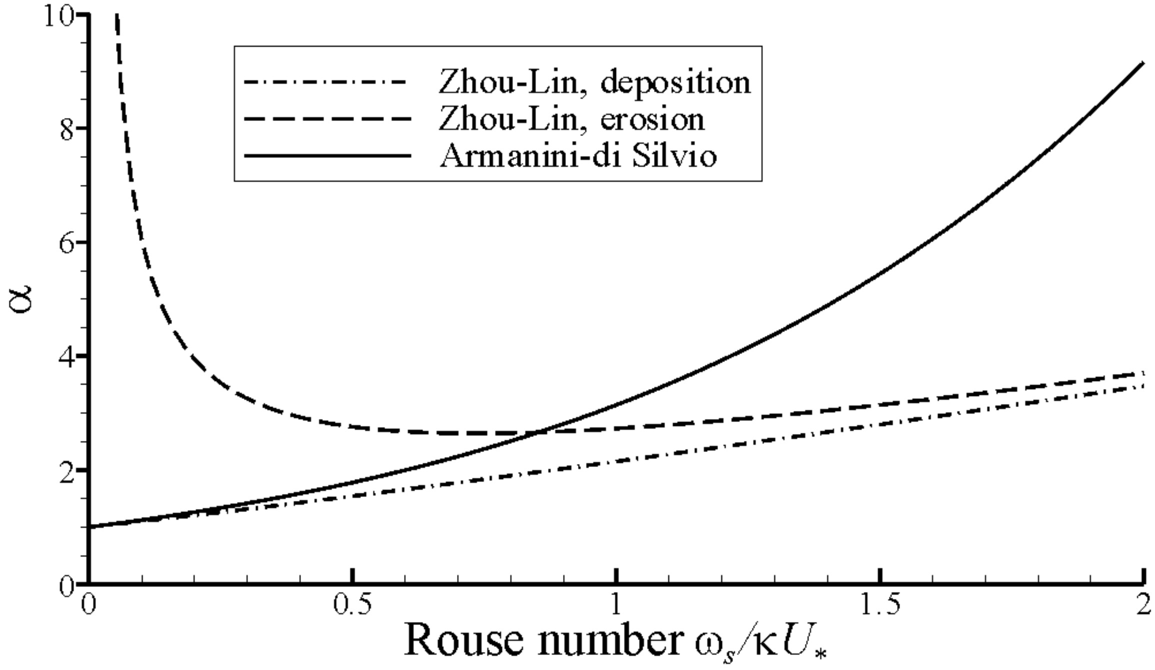
$$\alpha = \frac{R}{4} + \frac{\sigma_1^2}{R}$$

where $R = 6\omega_s/(\kappa U_*)$ and σ_1 is the first positive root of the following equations:

$$tg(\sigma) = -\frac{\sigma}{R} \qquad \text{for erosion}$$

$$2ctg(\sigma) = \frac{2\sigma}{R} - \frac{R}{2\sigma} \qquad \text{for deposition}$$

Comparison of Armanini and de Silvio's and Zhou and Lin's functions.



Both functions give $\alpha > 1$ theoretically.

Factors Affecting α:

- Cross-sectional shape
- Effect of sediment concentration on settling velocity
- Lift force (Saffman force)
- Bed forms

In 1-D models, α is often given 0.25 for strong deposition; 1.0 for strong erosion; and 0.5 for mild deposition and erosion (Han, 1980; Wu, 1991).

Calibration of α using measurement data is recommended for a specific case study.

Equilibrium Transport Model

Depth-averaged 2-D suspended-load transport equation

$$\frac{\partial}{\partial t}\left(\frac{hC}{\beta_s}\right)+\frac{\partial(hU_xC)}{\partial x}+\frac{\partial(hU_yC)}{\partial y}=\frac{\partial}{\partial x}\left[h\left(\varepsilon_s\frac{\partial C}{\partial x}+D_{sx}\right)\right]+\frac{\partial}{\partial y}\left[h\left(\varepsilon_s\frac{\partial C}{\partial y}+D_{sy}\right)\right]+E_b-D_b$$

2-D bed-load mass balance equation

$$(1-p'_m)\frac{\partial z_b}{\partial t}+\frac{\partial}{\partial t}\left(\frac{q_b}{u_b}\right)+\frac{\partial(\alpha_{bx}q_b)}{\partial x}+\frac{\partial(\alpha_{by}q_b)}{\partial y}=D_b-E_b$$

Three unknowns: C, q_b, and $\partial z_b/\partial t$, but there are only two equations. One equation is needed to close the model.

Equilibrium Transport Model of Bed Load

$$q_b=q_{b*}(U,h,\tau,d,\gamma_s,\cdots)$$

Bed Change Equations

For only suspended load

$$(1-p'_m)\frac{\partial z_b}{\partial t} = D_b - E_b = \alpha\omega_s(C - C_*)$$

For only bed load

$$(1-p'_m)\frac{\partial z_b}{\partial t} = \frac{1}{L_b}(q_b - q_{b*})$$

For total load

$$(1-p'_m)\frac{\partial z_b}{\partial t} = \frac{1}{L_t}(q_t - q_{t*}) \quad \text{or} \quad (1-p'_m)\frac{\partial z_b}{\partial t} = D_b - E_b + \frac{1}{L}(q_b - q_{b*})$$

Note that the difference between L_b, L_t, and L. L_b is the adaptation length of bed load. L_t is the adaptation length of total load. L is approximately equal to L_t in general and reduces to L_b in the case of bed load.

Non-equilibrium Bed-Load Transport

From

$$(1-p'_m)\frac{\partial z_b}{\partial t}+\frac{\partial}{\partial t}\left(\frac{q_b}{u_b}\right)+\frac{\partial(\alpha_{bx}q_b)}{\partial x}+\frac{\partial(\alpha_{by}q_b)}{\partial y}=D_b-E_b$$

$$(1-p'_m)\frac{\partial z_b}{\partial t}=D_b-E_b+\frac{1}{L}(q_b-q_{b*})$$

one can derive

$$\frac{\partial}{\partial t}\left(\frac{q_b}{u_b}\right)+\frac{\partial(\alpha_{bx}q_b)}{\partial x}+\frac{\partial(\alpha_{by}q_b)}{\partial y}=\frac{1}{L}(q_{b*}-q_b)$$

Thus, the sediment transport model is also closed.

2-D Non-equilibrium Transport Model

Approach I: Bed-Load/Suspended-Load Model

Depth-averaged 2-D suspended-load transport equation

$$\frac{\partial}{\partial t}\left(\frac{hC}{\beta_s}\right) + \frac{\partial(hU_x C)}{\partial x} + \frac{\partial(hU_y C)}{\partial y} = \frac{\partial}{\partial x}\left[h\left(\varepsilon_s \frac{\partial C}{\partial x} + D_{sx}\right)\right] + \frac{\partial}{\partial y}\left[h\left(\varepsilon_s \frac{\partial C}{\partial y} + D_{sy}\right)\right] + E_b - D_b$$

2-D bed-load transport equation

$$\frac{\partial}{\partial t}\left(\frac{q_b}{u_b}\right) + \frac{\partial(\alpha_{bx} q_b)}{\partial x} + \frac{\partial(\alpha_{by} q_b)}{\partial y} = \frac{1}{L}(q_{b*} - q_b)$$

Bed change equation

$$(1 - p'_m)\frac{\partial z_b}{\partial t} = D_b - E_b + \frac{1}{L}(q_b - q_{b*})$$

Approach II: Bed-Material Load Model

Depth-averaged 2-D bed-material (or total) load transport equation

$$(1-p'_m)\frac{\partial z_b}{\partial t}+\frac{\partial}{\partial t}\left(\frac{hC_t}{\beta_t}\right)+\frac{\partial q_{tx}}{\partial x}+\frac{\partial q_{ty}}{\partial y}=0$$

Sediment near-bed exchange equation

$$(1-p'_m)\frac{\partial z_b}{\partial t}=\frac{1}{L_t}(q_t-q_{t*})=\alpha_t\omega_s(C_t-C_{t*})$$

Combining the above equations yields

$$\frac{\partial}{\partial t}\left(\frac{hC_t}{\beta_t}\right)+\frac{\partial(hU_xC_t)}{\partial x}+\frac{\partial(hU_yC_t)}{\partial y}=\frac{\partial}{\partial x}\left[h\left(\varepsilon_s\frac{\partial(r_sC_t)}{\partial x}+D_{sx}\right)\right]+\frac{\partial}{\partial y}\left[h\left(\varepsilon_s\frac{\partial(r_sC_t)}{\partial y}+D_{sy}\right)\right]+\alpha_t\omega_s(C_{t*}-C_t)$$

The last two equations are the governing equations for model approach II. They are closed by assuming $r_s=q_{s*}/q_{t*}$ and relations for β_t and α_t.

Adaptation Length of Sediment

For suspended load

$$L_s = \frac{Uh}{\alpha \omega_s}$$

For bed load, L_b is closely related to bed forms, and assumed to take the length of the dominant bed forms.

For bed-material load

$$L_t = \max\{L_b,\ L_s\} \qquad \text{or} \qquad L_t = (1 - r_s)L_b + r_s L_s$$

This option gives more stable solution

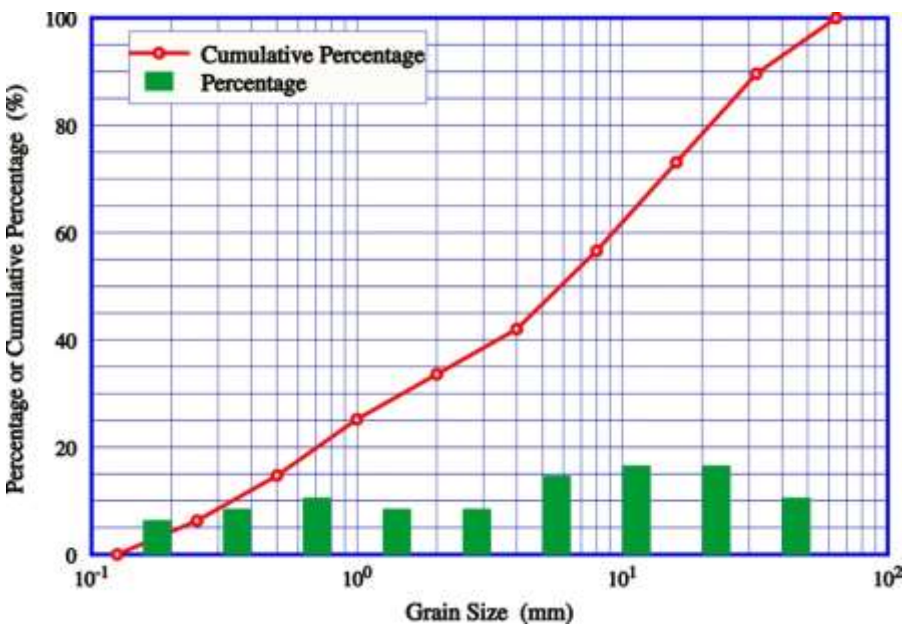
For wash load, α is set as zero and L_t is given infinitely large.

Calibration of L_b and L_t using site specific data is recommended.

Non-uniform Sediment Transport

In the case of low sediment concentration, the interactions among the moving sediment particles are usually negligible, so that each size class of the moving sediment mixture can be assumed to have the same transport behavior as uniform sediment.

Therefore, the sediment mixture is divided to *N* size classes.

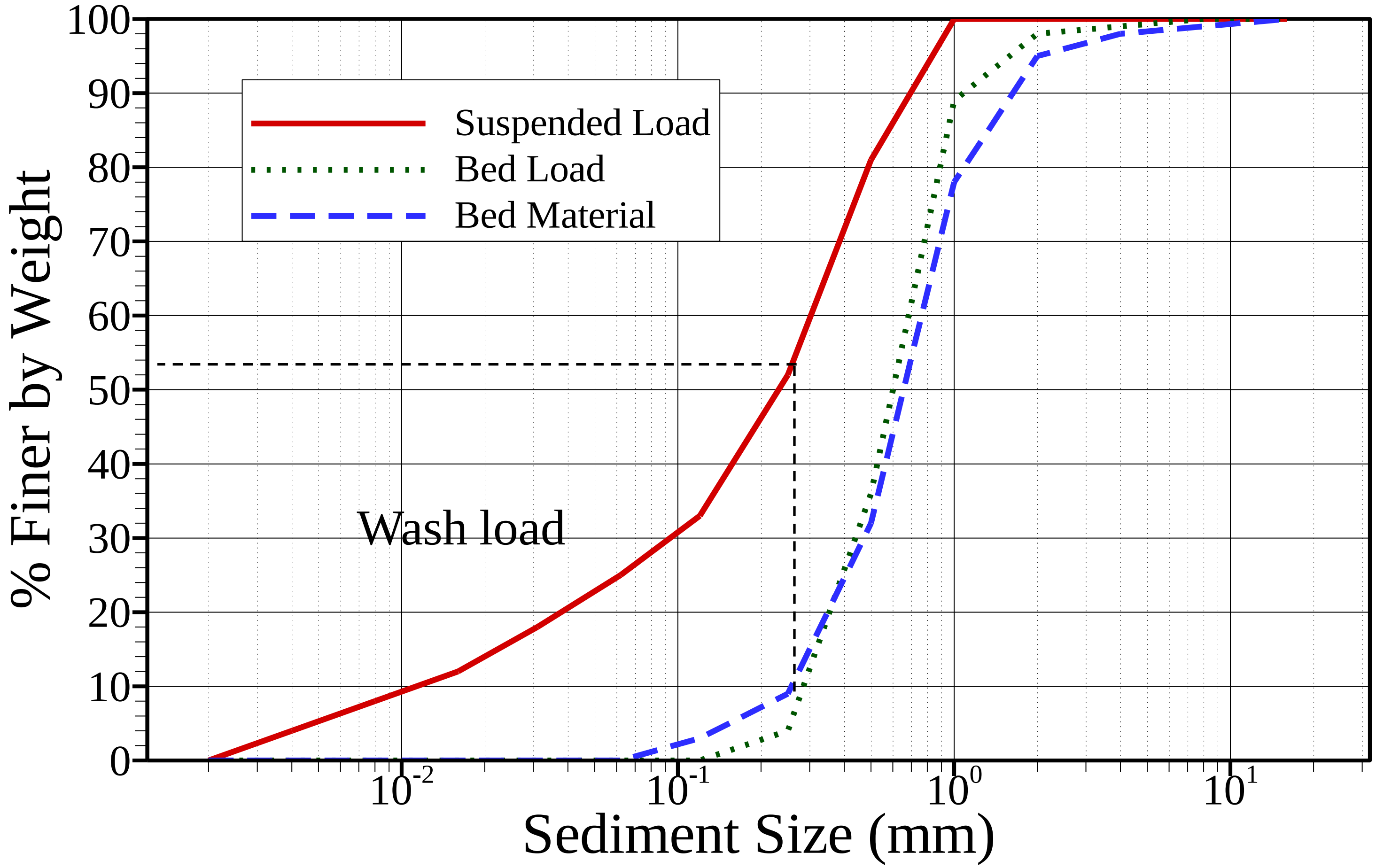


How many classes are enough?

- Suitable to the study case
- Restricted by measurement accuracy
- Restricted by computational efficiency

Size classes cover all bed load, suspended load, bed material, and bank material

Each size class has bed load and suspended load



100
90
80
70
60
50
40
30
20
10
0
% Finer by Weight
Suspended Load
Bed Load
Bed Material
Wash load
10-2
10-1
100
101
Sediment Size (mm)

Representative Size of Each Class

Given the lower and upper bounds of sediment size of each size class, the representative size is determined as

$$d_i = \sqrt{d_{i,lower} d_{i,upper}}$$

$$d_i = \left(d_{i,lower} + d_{i,upper}\right)/2$$

or

$$d_i = \left(d_{i,lower} + d_{i,upper} + \sqrt{d_{i,lower} d_{i,upper}}\right)/3$$

The transport of the kth size class of suspended load is described by

$$\frac{\partial}{\partial t}\left(\frac{hC_k}{\beta_{sk}}\right)+\frac{\partial(hU_x C_k)}{\partial x}+\frac{\partial(hU_y C_k)}{\partial y}=\frac{\partial}{\partial x}\left[h\left(\varepsilon_s\frac{\partial C_k}{\partial x}+D_{sxk}\right)\right]+\frac{\partial}{\partial y}\left[h\left(\varepsilon_s\frac{\partial C_k}{\partial y}+D_{syk}\right)\right]+\alpha\omega_{sk}(C_{*k}-C_k)\quad (k=1,\,2,\,\ldots,\,N)$$

The transport of the kth size class of bed load:

$$\frac{\partial}{\partial t}\left(\frac{q_{bk}}{u_{bk}}\right)+\frac{\partial(\alpha_{bx}q_{bk})}{\partial x}+\frac{\partial(\alpha_{by}q_{bk})}{\partial y}=\frac{1}{L}(q_{b*k}-q_{bk})\quad (k=1,\,2,\,\ldots,\,N)$$

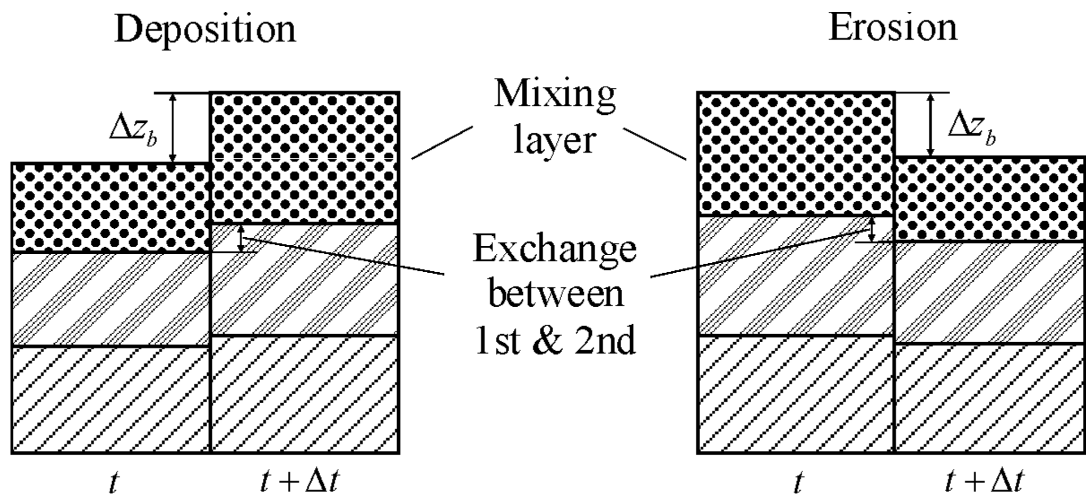
The fractional bed change is determined by

$$(1-p'_m)\left(\frac{\partial z_b}{\partial t}\right)_k=\alpha\omega_{sk}(C_k-C_{*k})+\frac{1}{L}(q_{bk}-q_{b*k})\quad (k=1,\,2,\,\ldots,\,N)$$

The total bed change is determined by $\displaystyle\frac{\partial z_b}{\partial t}=\sum_{k=1}^{N}\left(\frac{\partial z_b}{\partial t}\right)_k$

Bed Material Sorting

The bed material is divided to an adequate number of layers. The first (surface) layer is called the mixing layer.



The change in size composition of the mixing layer is determined by

$$\frac{\partial(\delta_m p_{bk})}{\partial t} = \left(\frac{\partial z_b}{\partial t}\right)_k + p_{bk}^* \left(\frac{\partial \delta_m}{\partial t} - \frac{\partial z_b}{\partial t}\right) \qquad (k = 1, 2, \ldots, N)$$

where δ_m is the mixing layer thickness; p_{bk} is the size composition.

Bed Material Sorting (cont'd)

If no exchange between the second and third layers, the change in size composition of the second layer is determined by

$$\frac{\partial(\delta_s p_{sbk})}{\partial t} = -p_{bk}^{*}\left(\frac{\partial \delta_m}{\partial t} - \frac{\partial z_b}{\partial t}\right)$$

where p_{sbk} is the fraction of the *k*th size class of bed material contained in the second layer, δ_s is the thickness of the second layer, and

$$p_{bk}^{*} = \begin{cases} p_{bk} & \text{if } \partial z_b/\partial t - \partial \delta_m/\partial t \geq 0 \\ p_{sbk} & \text{if } \partial z_b/\partial t - \partial \delta_m/\partial t < 0 \end{cases}$$

Mixing Layer Thickness

The thickness of the mixing layer is related to the time scale under consideration. For very short, intermediate, and long time, it may be at the order of sediment size, bed form height, and bed deformation thickness, respectively.

Available methods for mixing layer thickness:

$$\delta_m = (0.1 \sim 0.2)\, h$$ (Karim and Kennedy, 1982)

$$\delta_m = \frac{d_L}{(1 - p'_m)\, p_{bm}}$$ (Borah et al., 1982)

$$\delta_m = 2 d_{50} \frac{\tau'_b}{\tau_{c50}}$$ (Van Niekerk et al., 1992)

$$\delta_m = \max\left[0.5\Delta,\ 2 d_{50}\right]$$ (Wu and Vieira, 2002)

3-D Sediment Transport Model

Suspended-load transport equation

$$\frac{\partial c}{\partial t}+\frac{\partial(u_x c)}{\partial x}+\frac{\partial(u_y c)}{\partial y}+\frac{\partial(u_z c)}{\partial z}-\frac{\partial(\omega_s c)}{\partial z}=\frac{\partial}{\partial x}\left(\varepsilon_s\frac{\partial c}{\partial x}\right)+\frac{\partial}{\partial y}\left(\varepsilon_s\frac{\partial c}{\partial y}\right)+\frac{\partial}{\partial z}\left(\varepsilon_s\frac{\partial c}{\partial z}\right)$$

At the interface between bed load and suspended load:

$$E_b=-\varepsilon_s\frac{\partial c}{\partial z}\bigg|_{z=z_b+\delta}=\omega_s c_{b*} \qquad D_b=\omega_s c_b$$

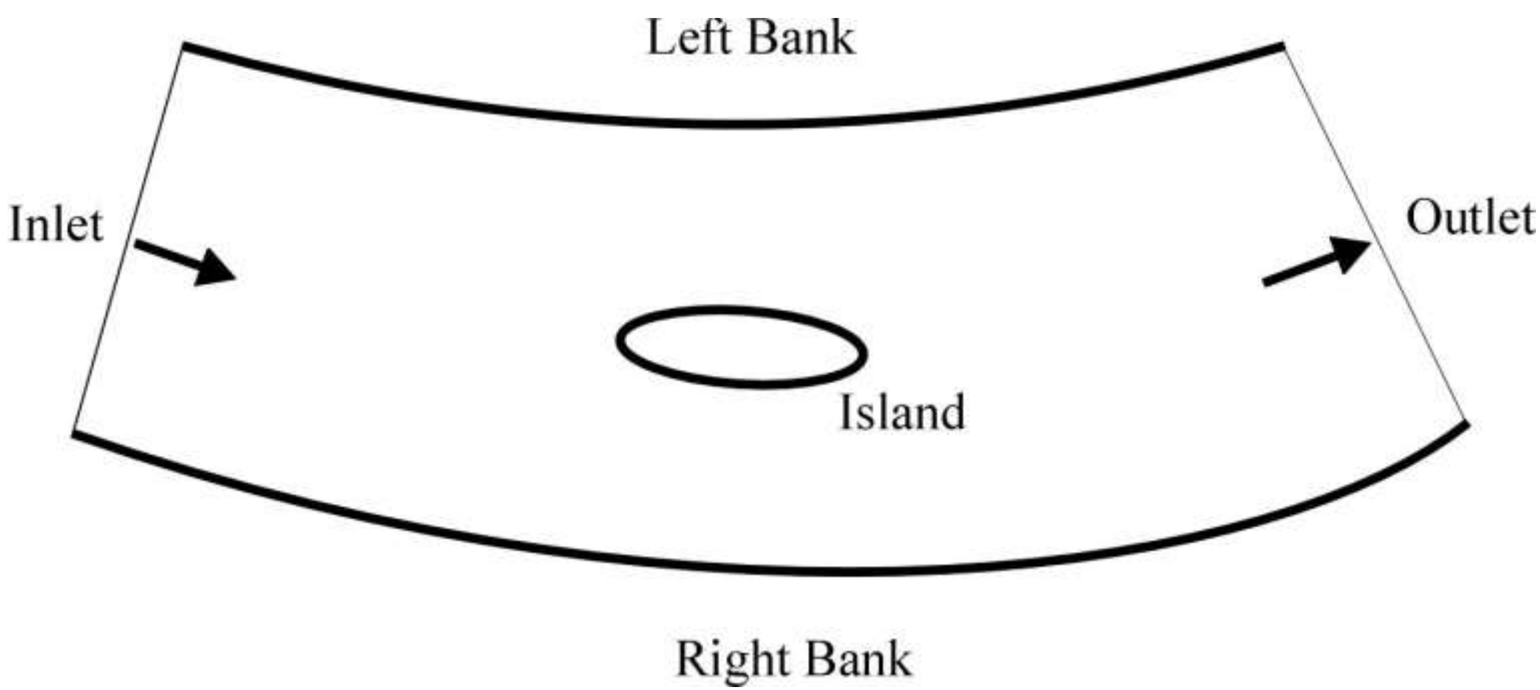
Bed-load transport equation

$$\frac{\partial}{\partial t}\left(\frac{q_b}{u_b}\right)+\frac{\partial(\alpha_{bx}q_b)}{\partial x}+\frac{\partial(\alpha_{by}q_b)}{\partial y}=\frac{1}{L}(q_{b*}-q_b)$$

Bed change equation

$$(1-p'_m)\frac{\partial z_b}{\partial t}=D_b-E_b+\frac{1}{L}(q_b-q_{b*})$$

Computational Conditions



1. Initial Bed Material Gradation
2. Time Series of Inflow Sediment Discharge
3. Inflow Sediment Gradation

Boundary Conditions

Inlet

$$q_{bk} = \frac{Q_{bk} q^{m_b}}{\int_0^B q^{m_b} dy} \qquad q_{sk} = \frac{Q_{sk} q^{m_s}}{\int_0^B q^{m_s} dy}$$

Wall Boundaries

$$q_{bk} = 0 \qquad \frac{\partial C_k}{\partial n} = 0$$

Outlet

$$\frac{\partial C_k}{\partial s} = 0$$

1-D Non-equilibrium Transport Model

Sediment Transport Equation

$$\frac{\partial\left(AC_t\right)}{\partial t}+\frac{\partial Q_t}{\partial x}+\frac{1}{L}\left(Q_t-Q_{t*}\right)=q_t$$

Bed Change Equation

$$\left(1-p_m'\right)\frac{\partial A_b}{\partial t}=\frac{1}{L}\left(Q_t-Q_{t*}\right)$$

1-D Equilibrium Transport Model

Local Equilibrium Assumption

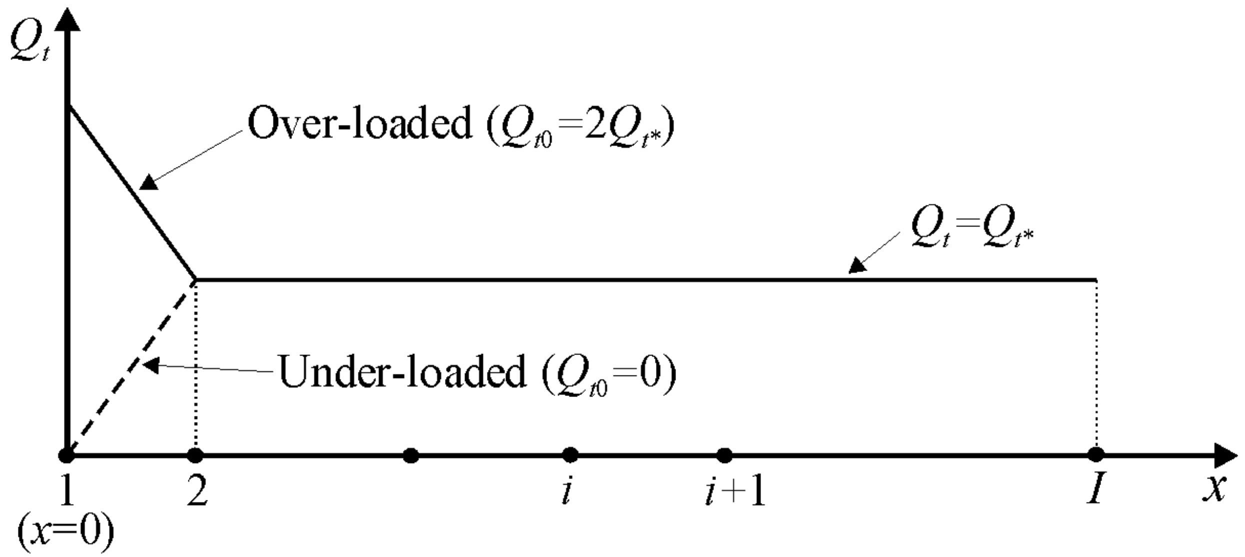
$$Q_t = Q_{t*}\left(U, h, d_{50}, B, \gamma_s, \gamma, \ldots\right)$$

Mass Balance – Exner Equation

$$\left(1 - p'_m\right)\frac{\partial A_b}{\partial t} + \frac{\partial Q_t}{\partial x} = q_t$$

Sediment Overloading

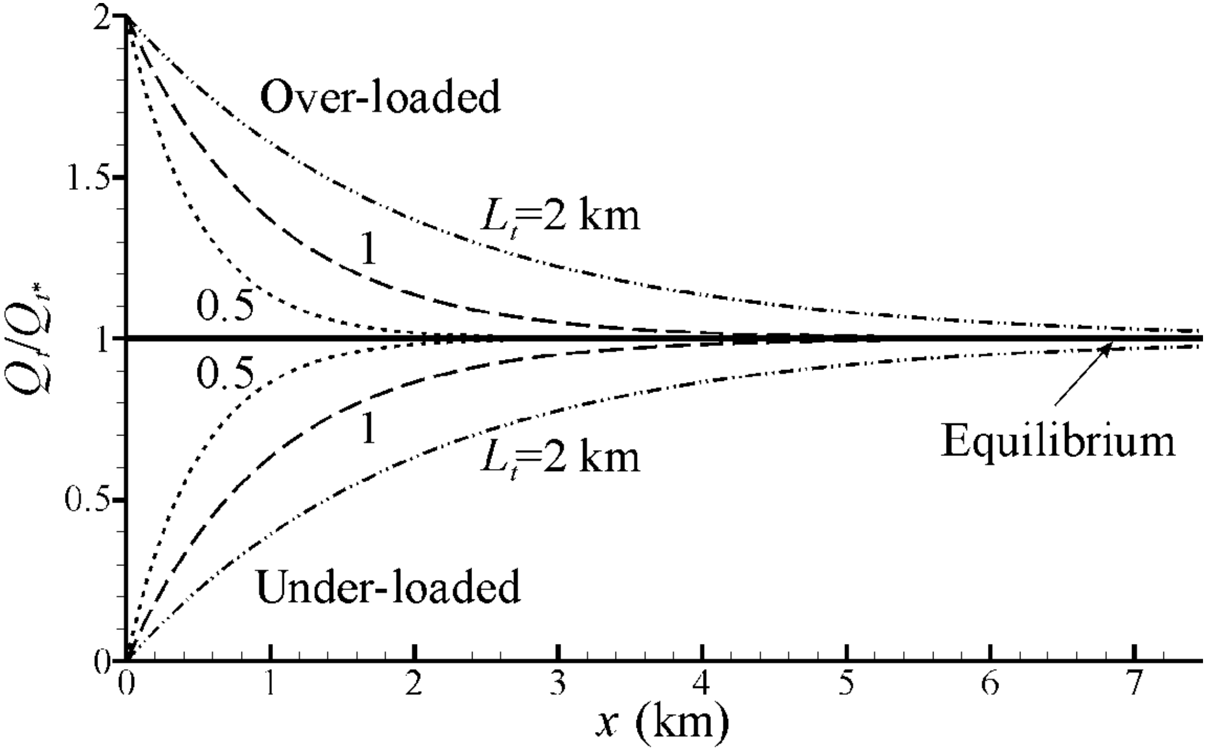
Equil. Model



$$Q_t = Q_{t*}\left(U, h, d_{50}, B, \gamma_s, \gamma, \ldots\right)$$

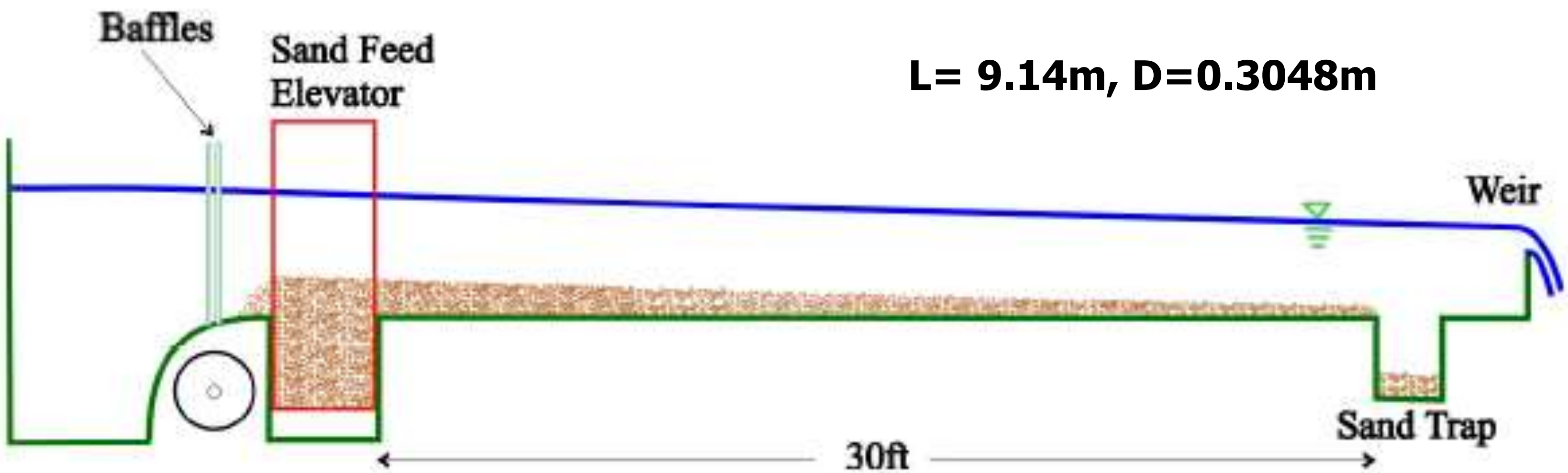
$$\left(1 - p'_m\right)\frac{\partial A_b}{\partial t} + \frac{\partial Q_t}{\partial x} = q_t$$

Non-Equil. Model



$$\frac{\partial Q_t}{\partial x} + \frac{1}{L}\left(Q_t - Q_{t*}\right) = 0$$

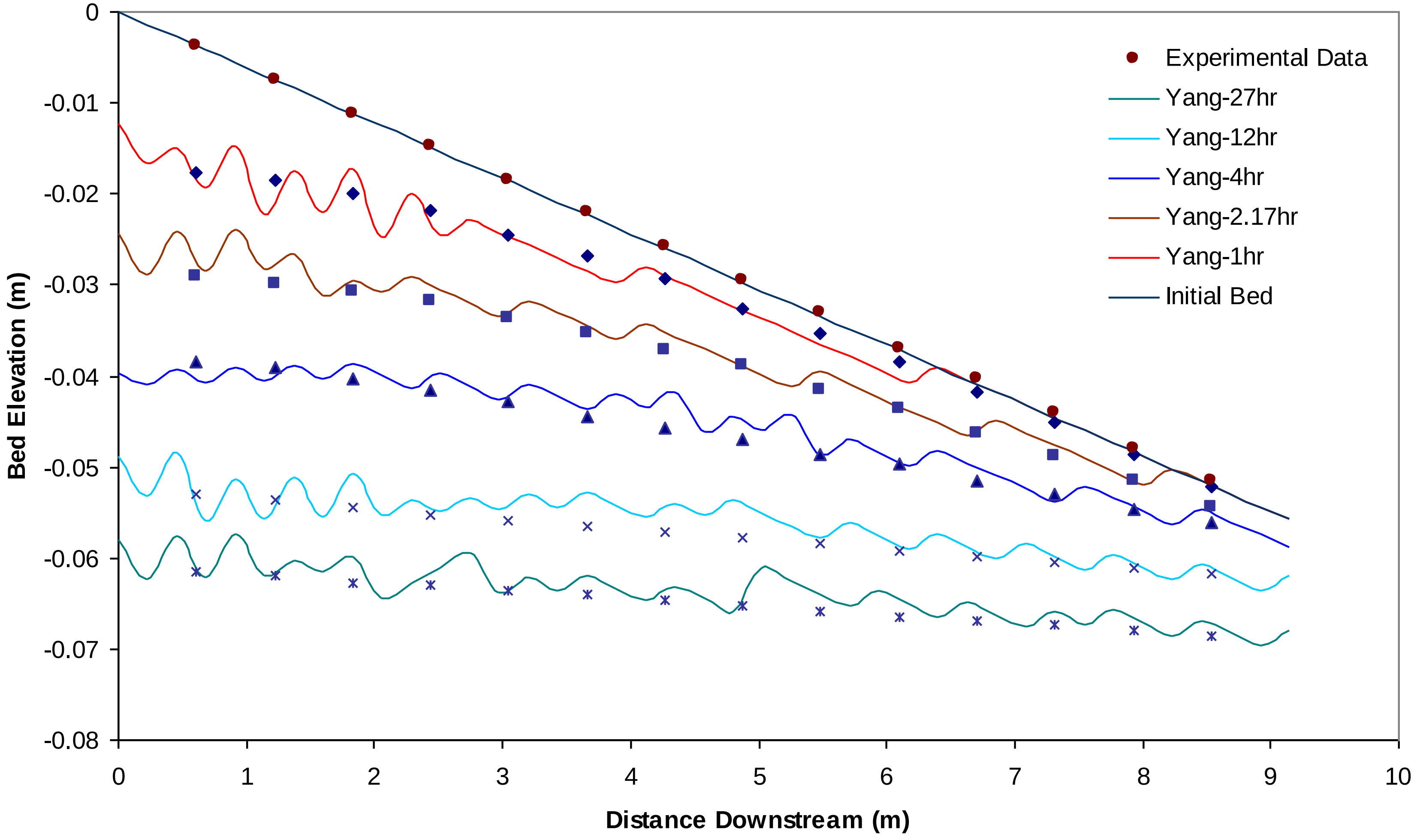
Channel Degradation (Newton, 1951)



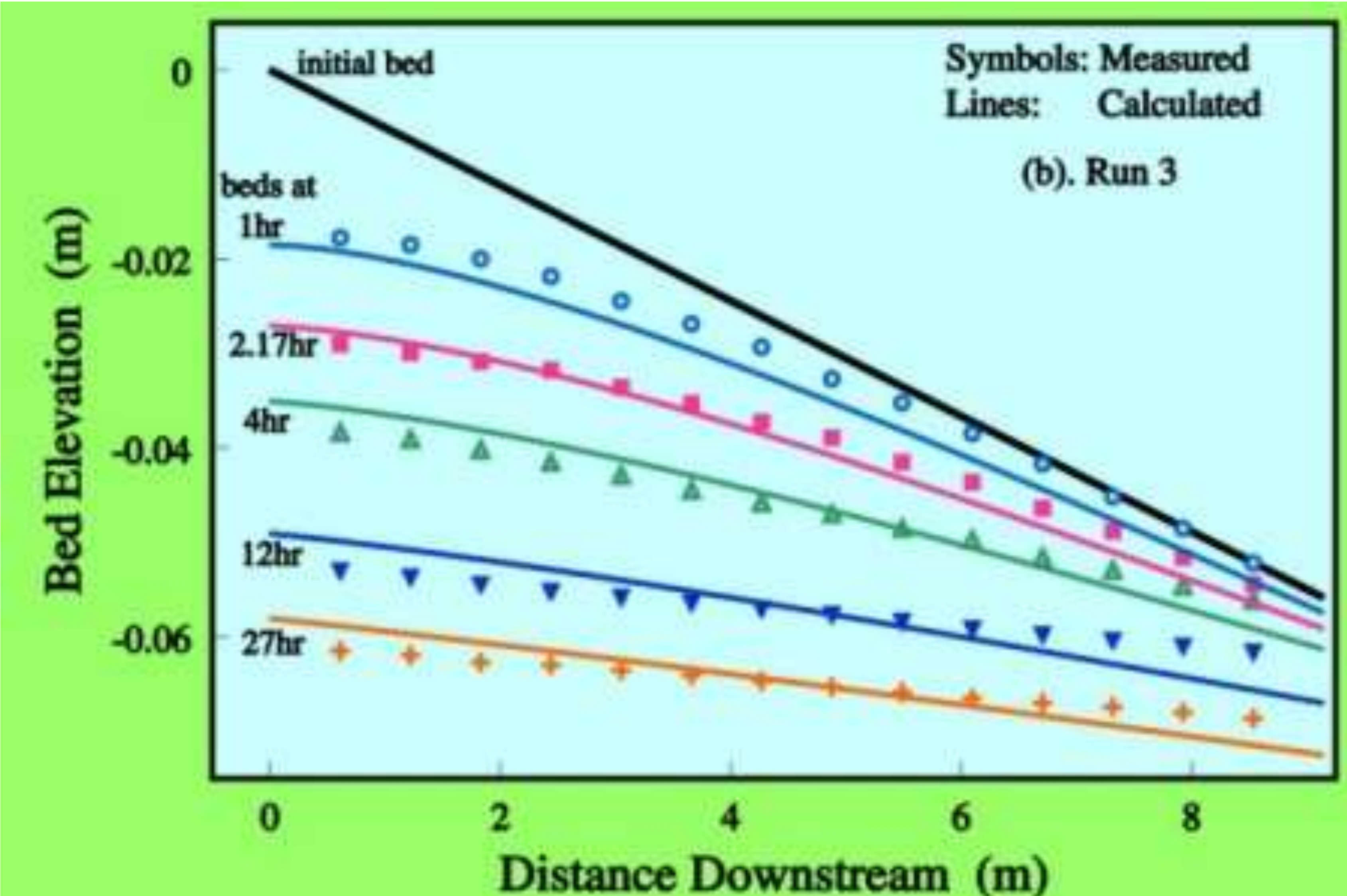
L= 9.14m, D=0.3048m

Exp.	Flow Discharge (m^3/s)	Sediment Size (mm)	Initial Bed Slope (m/m)	Initial n_b	Final n_b	Duration (hour)
Run 3	0.00566	0.69	0.0061	0.016	0.012	27

Simulation using an Equilibrium Model



Simulation with a Non-equilibrium Model



Summary of Non-Equilibrium Transport Model

- Well posed mathematically.
- Considers the temporal and spatial lags between flow and sediment transport;
- Easily handle the constrained sediment loading problem (strongly over- or under-loading);
- Easily handle the hard bottom problem;
- Calculate bed load and suspended load separately or combine them as bed-material load;
- Calculate wash load and bed-material load using a unified transport equation;
- More stable than the traditional equilibrium transport model.

Publications Related

W. Wu (2007), Computational River Dynamics, Taylor & Francis, UK, 494 p.